Modelling the human anatomy using Gaussian processes

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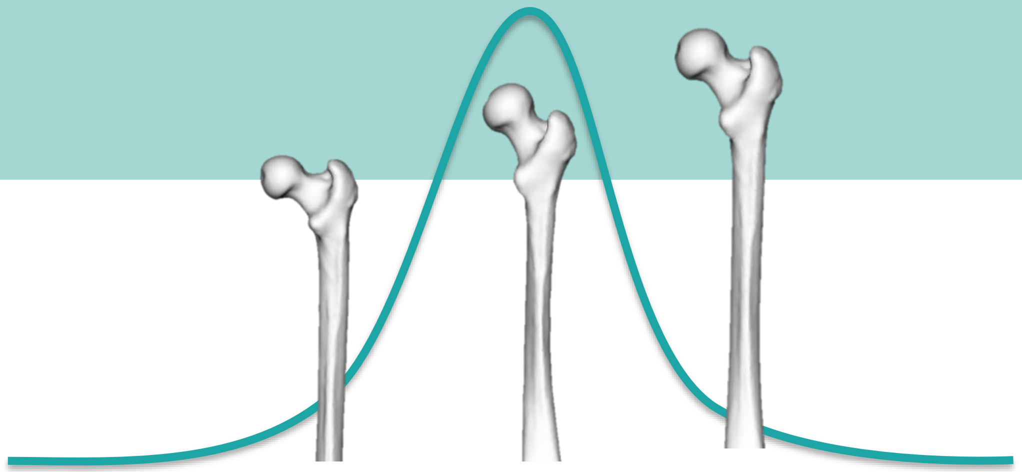
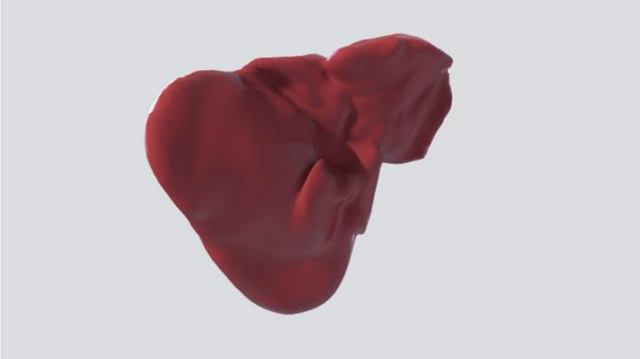
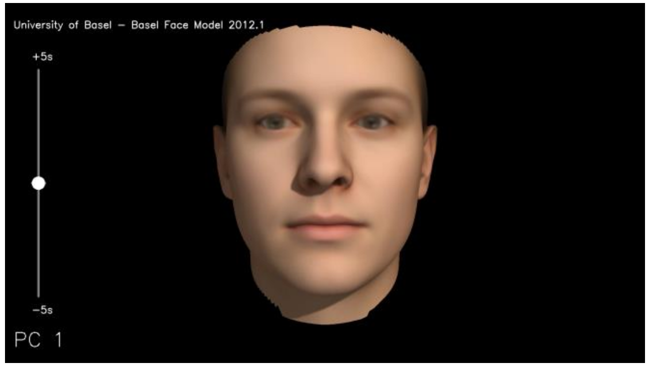
Special Interest Group on Gaussian Processes – 9. June 2021



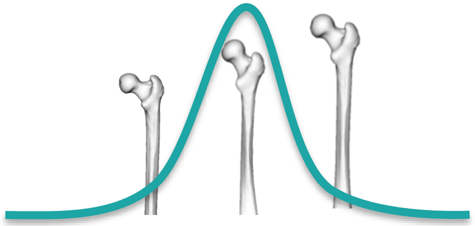
Statistical shape models



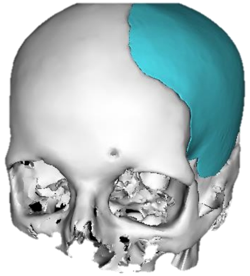


Animations: Jasenko Zivanov

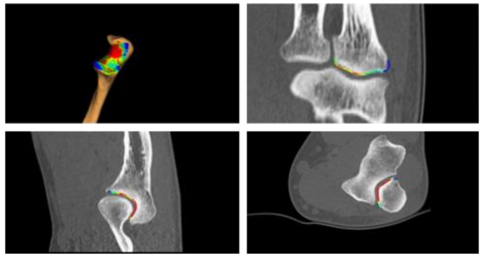
Applications of Statistical Shape Models



Data generation
- Testing on representative population data
- Training data generation for deep learning



Implant design
- Find best fitting shape to match anatomy

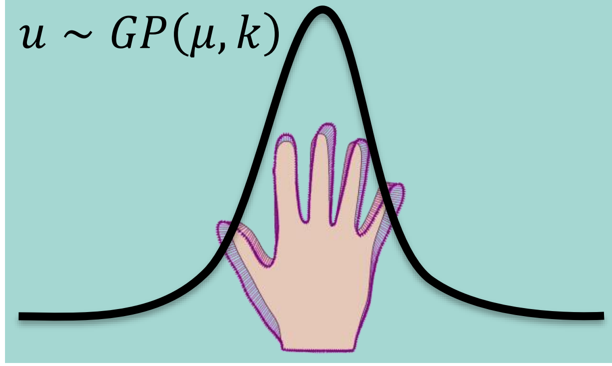


Shape and Image analysis
- Computer aided diagnosis
- Surgery planning
- Statistical inferences on populations

Outline

Introduction

- Shapes and shape models



Main part

- Modelling shape variations using Gaussian processes
- Inference using the Analysis-by-synthesis paradigm

Demo application

- Designing a patient specific implant

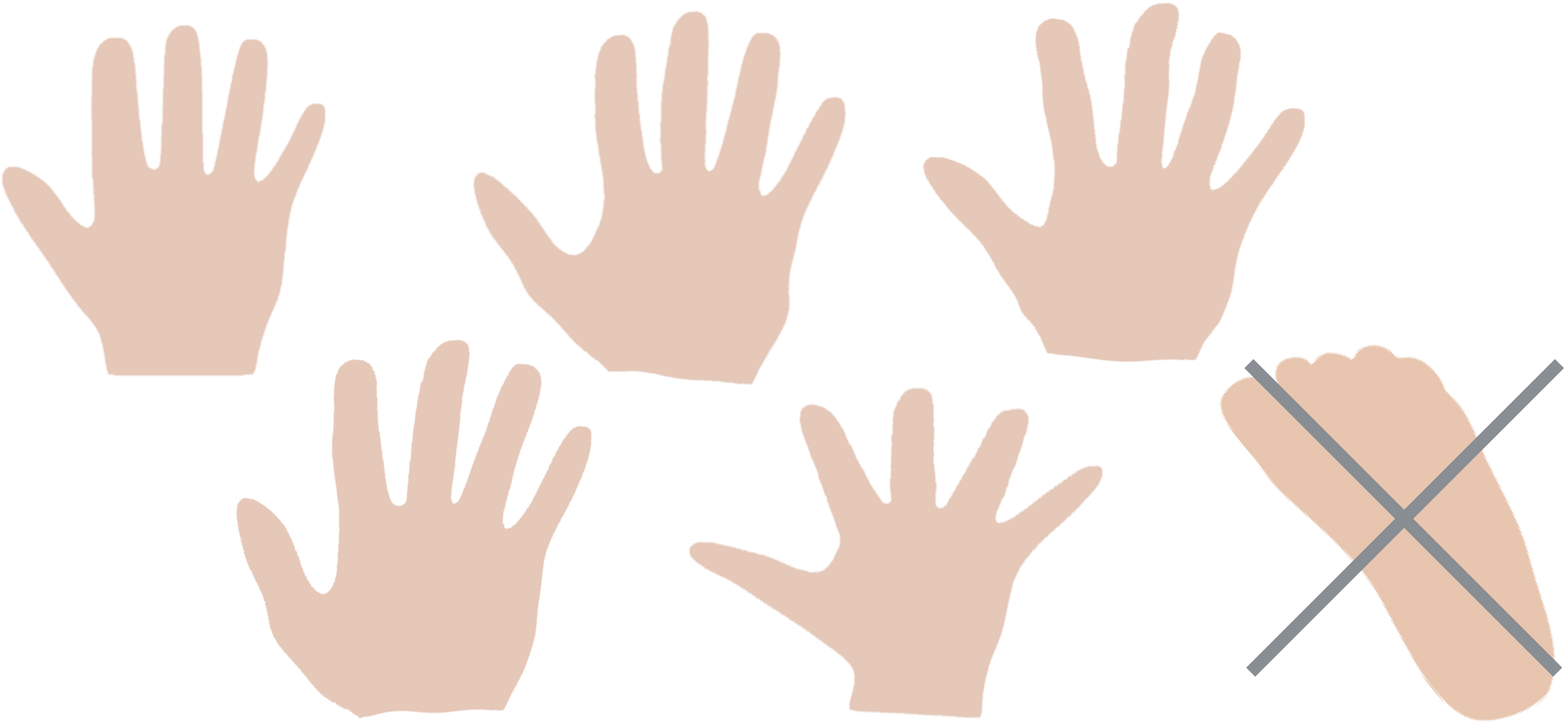
What is a shape?

Classical definition

All geometrical information that remains when location, scale and rotational effects are filtered out from an object.

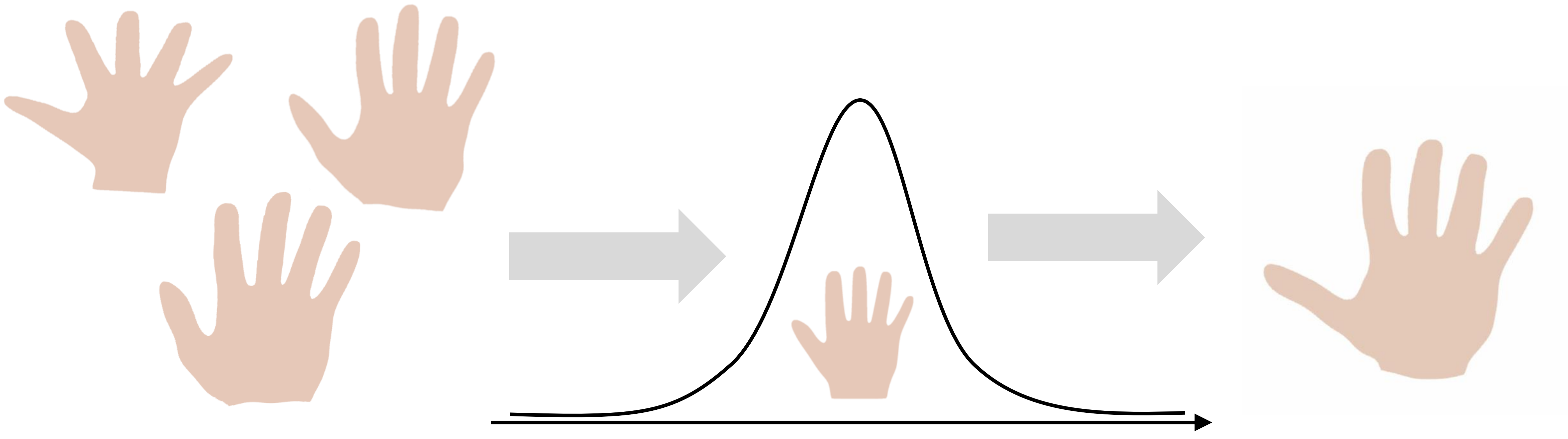
Shape families

Shape families

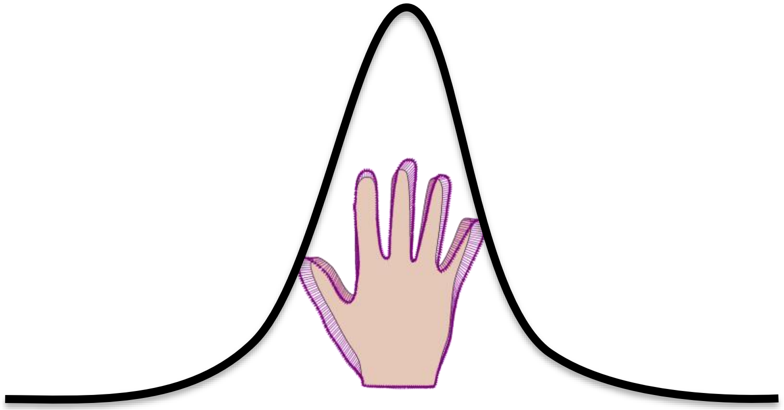


Shape families: Statistical shape models

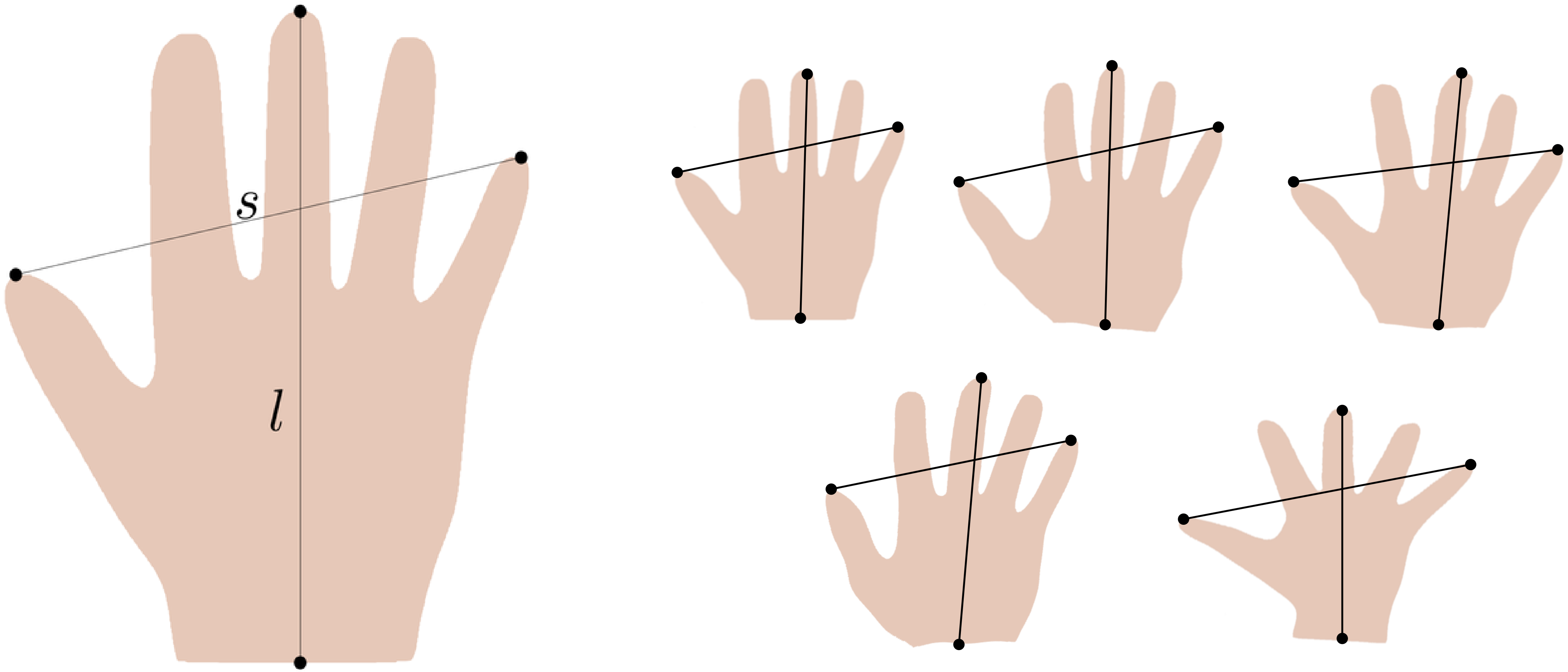
Example shapes



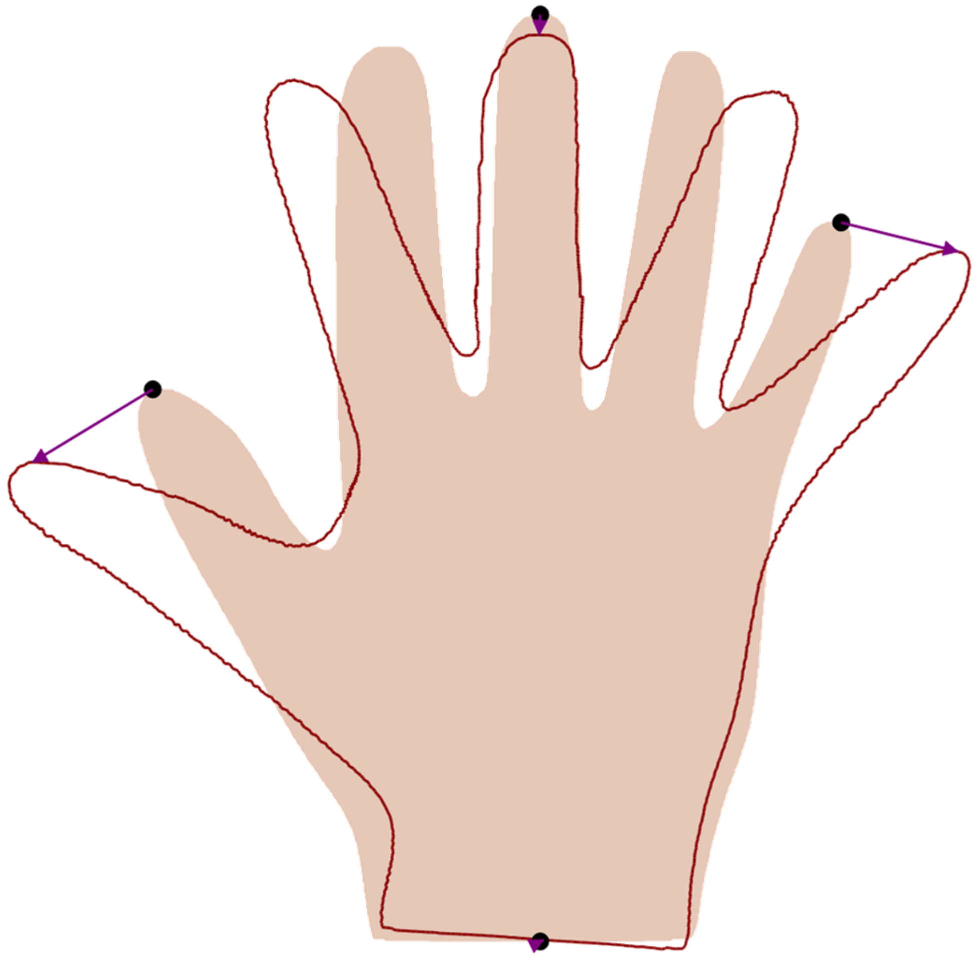
Modelling shape variations using Gaussian processes



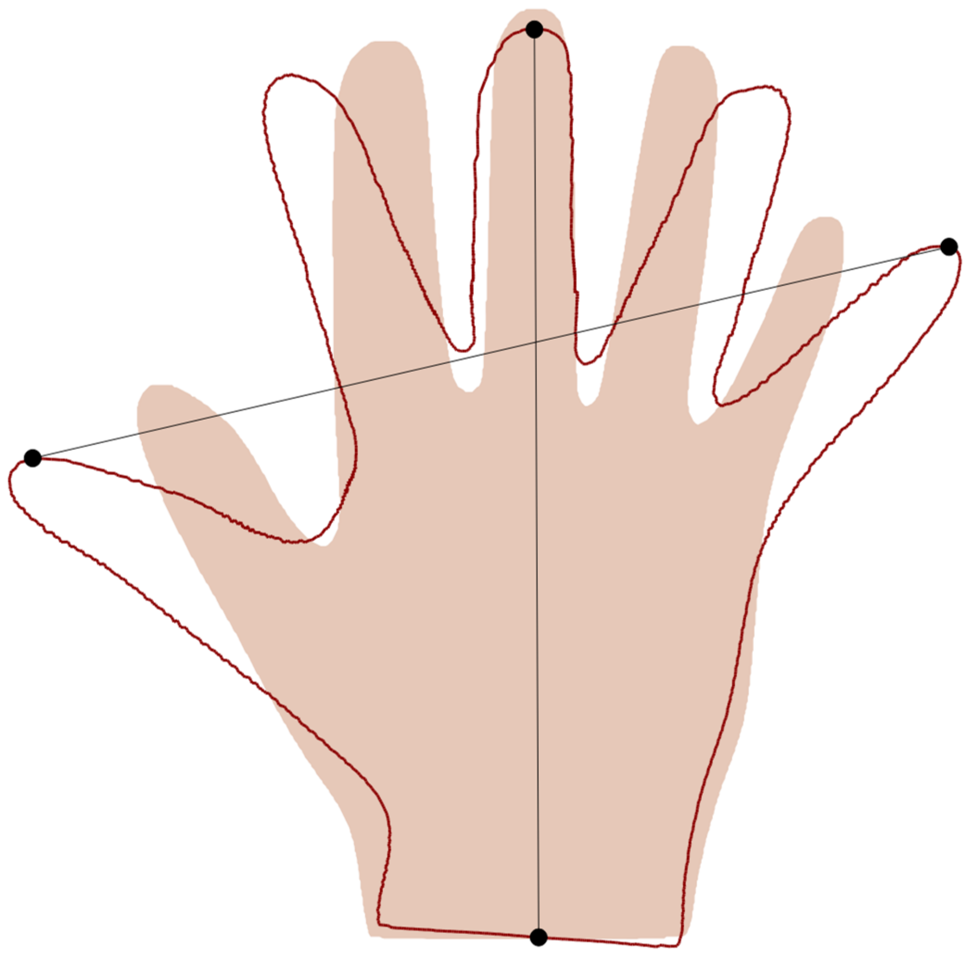
Measurements



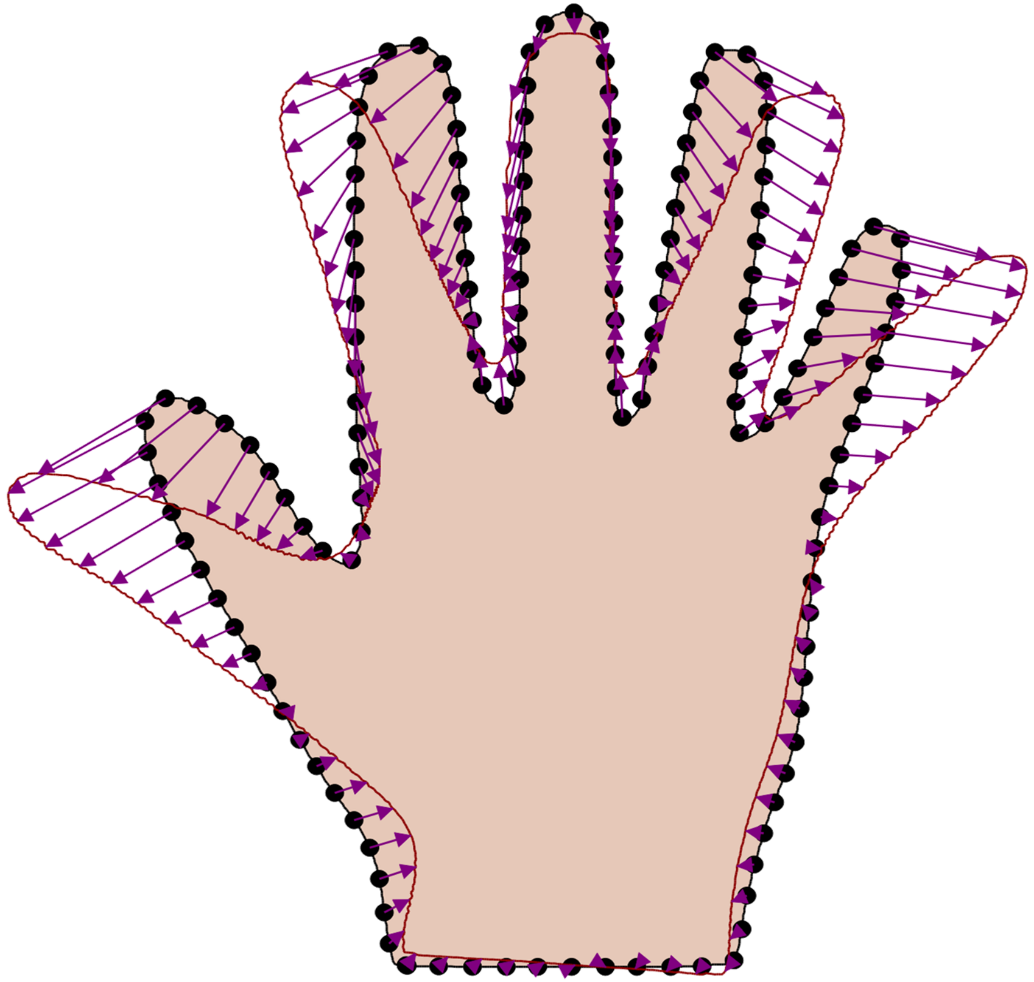
Shape changes as deformations



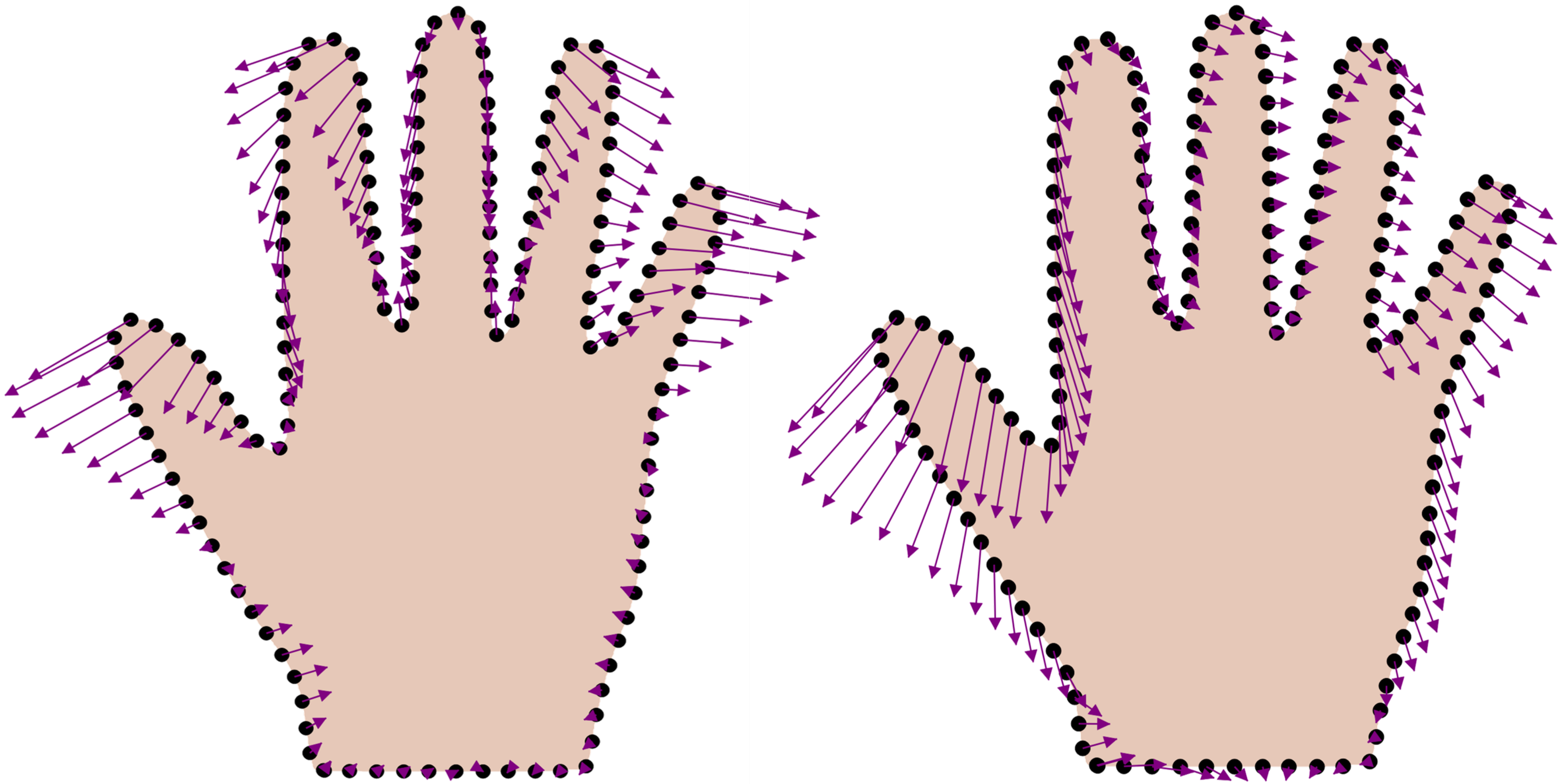
Shape changes as deformations



Shape changes as deformations



Shape changes as deformations



Shape changes as deformations

Set of points defining a reference shape:

$$\Gamma_R = \{x \mid x \in \mathbb{R}^2\}$$

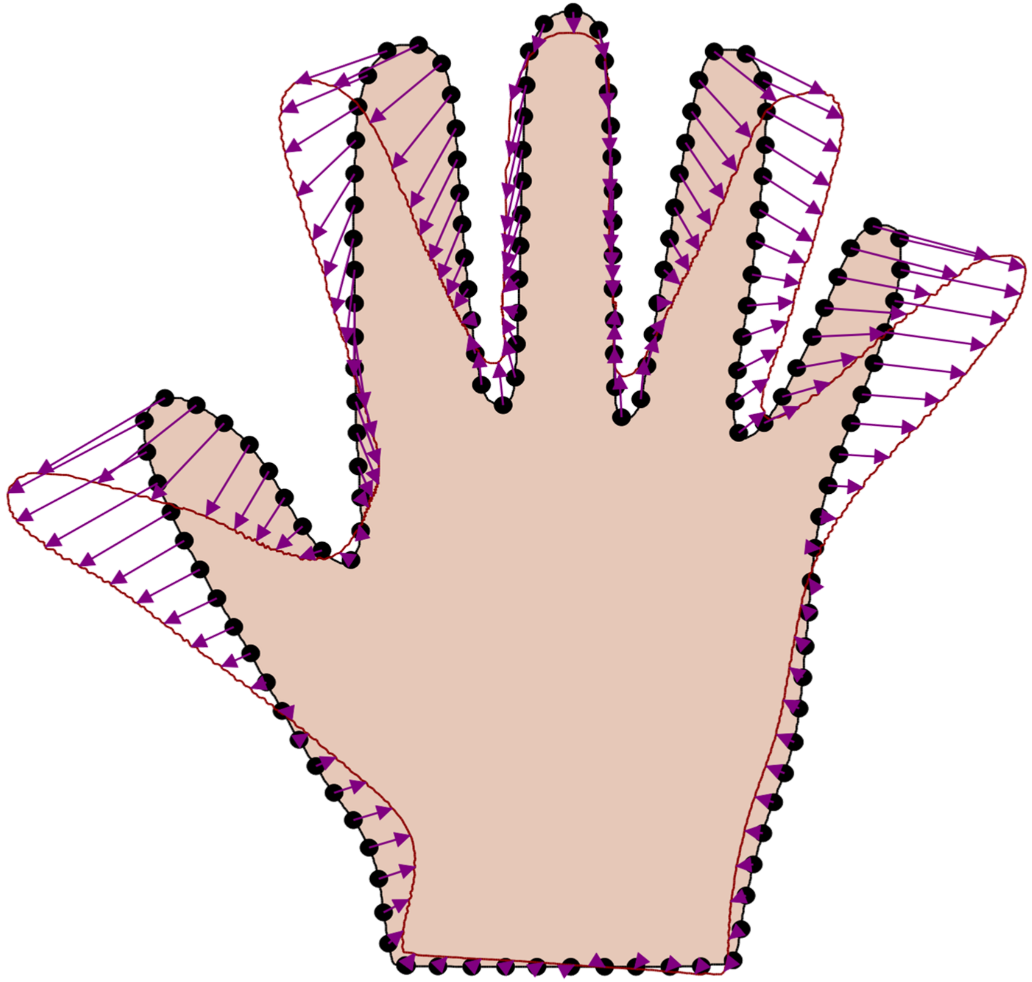
Vector field modelling the deformations

$$u : \Gamma_R \rightarrow \mathbb{R}^2$$

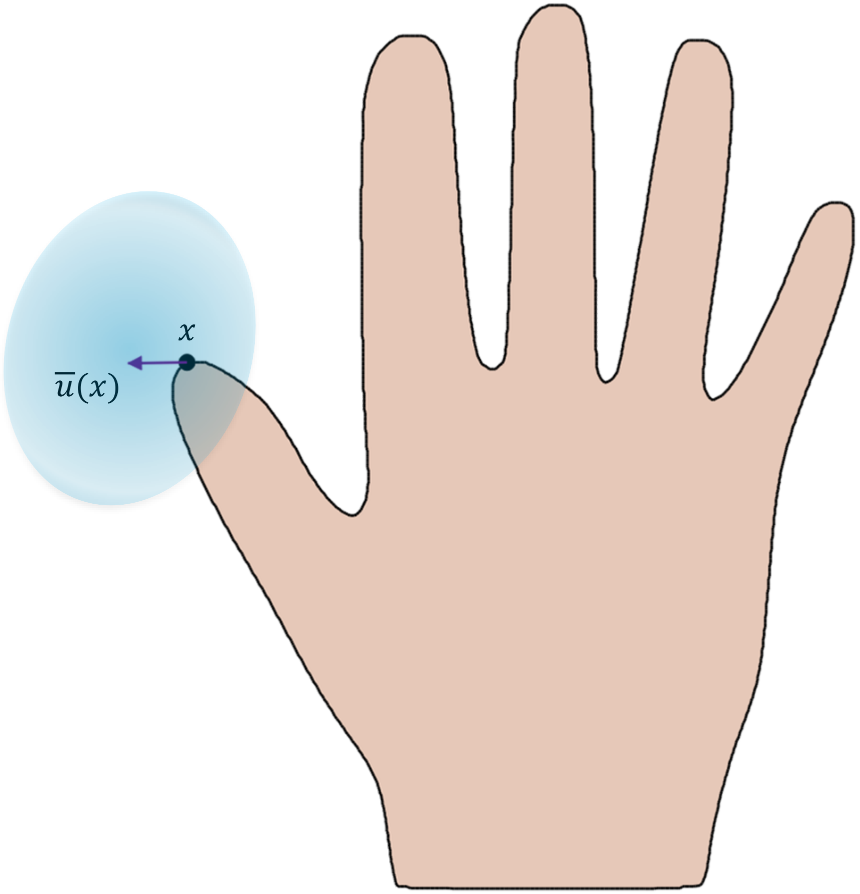
Shape defined by deformation field u

$$\Gamma = \{x + u(x) \mid x \in \Gamma_R\}$$

Our task: Model plausible deformation fields u!

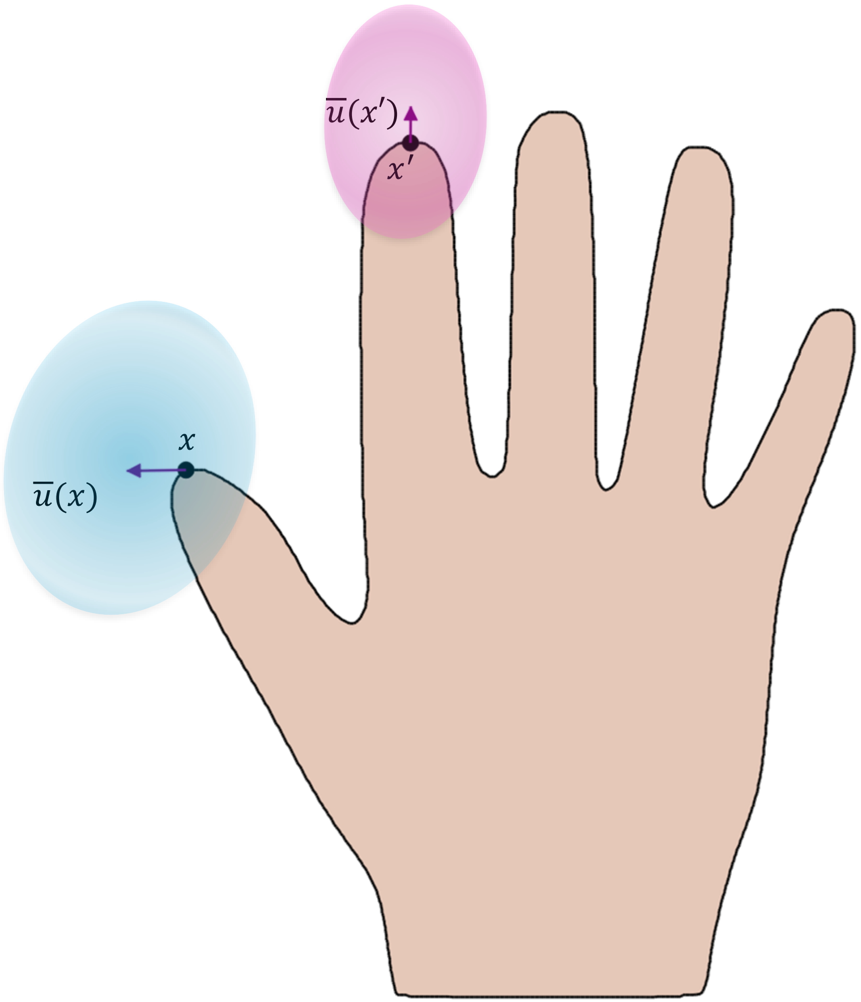


Modelling possible deformations



$$u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$$
$$\sim N\left(\begin{pmatrix} \overline{u}_1(x) \\ \overline{u}_2(x) \end{pmatrix}, \begin{pmatrix} \Sigma_{11}(x) & \Sigma_{12}(x) \\ \Sigma_{21}(x) & \Sigma_{22}(x) \end{pmatrix}\right)$$

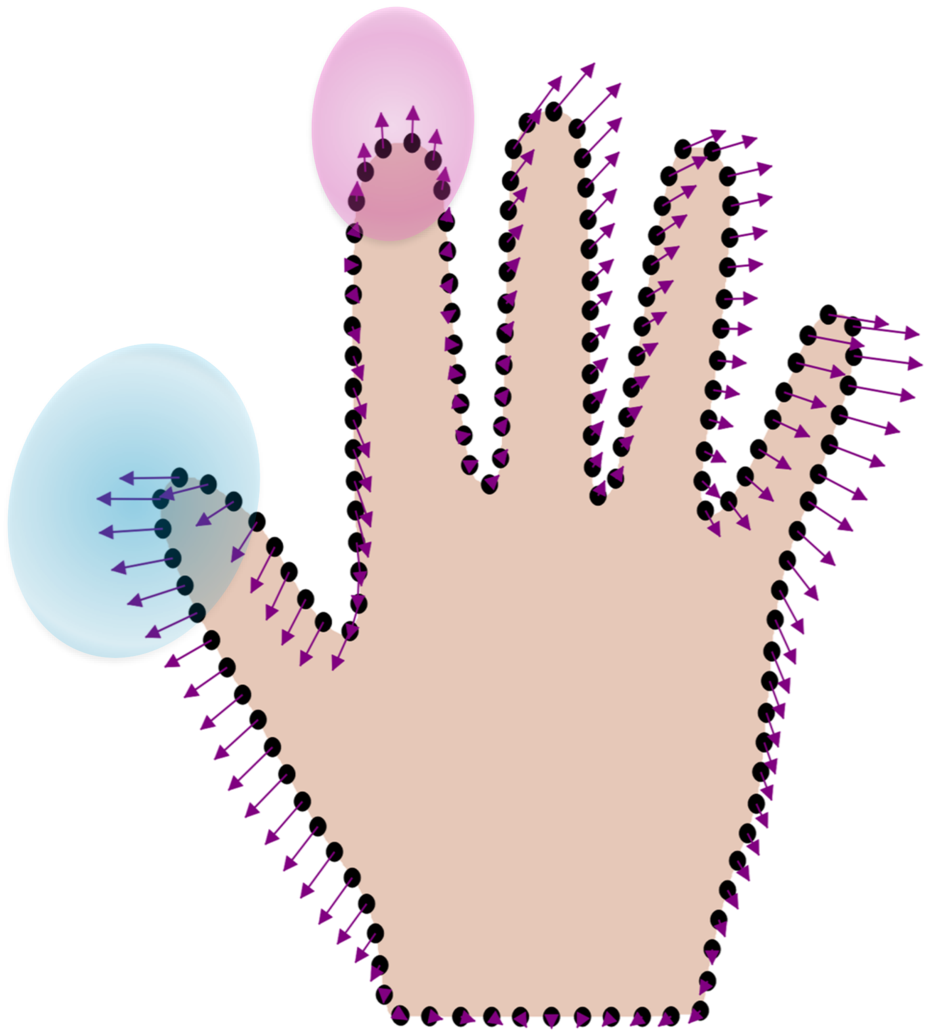
Modelling possible deformations



$$\begin{pmatrix} u(x) \\ u(x') \end{pmatrix} = \begin{pmatrix} u_1(x) \\ u_2(x) \\ u_1(x') \\ u_2(x') \end{pmatrix} \sim$$

$$N\left(\begin{pmatrix} \bar{u}_1(x) \\ \bar{u}_2(x) \\ \bar{u}_1(x') \\ \bar{u}_2(x') \end{pmatrix}, \begin{pmatrix} \Sigma_{11}(x,x) & \Sigma_{12}(x,x) & \Sigma_{11}(x,x') & \Sigma_{12}(x,x') \\ \Sigma_{21}(x,x) & \Sigma_{22}(x,x) & \Sigma_{21}(x,x') & \Sigma_{22}(x,x') \\ \Sigma_{11}(x',x) & \Sigma_{12}(x',x) & \Sigma_{11}(x',x') & \Sigma_{12}(x',x') \\ \Sigma_{21}(x',x) & \Sigma_{22}(x',x) & \Sigma_{21}(x',x') & \Sigma_{22}(x',x') \end{pmatrix} \right)$$

Modelling possible deformations



Idea

Define

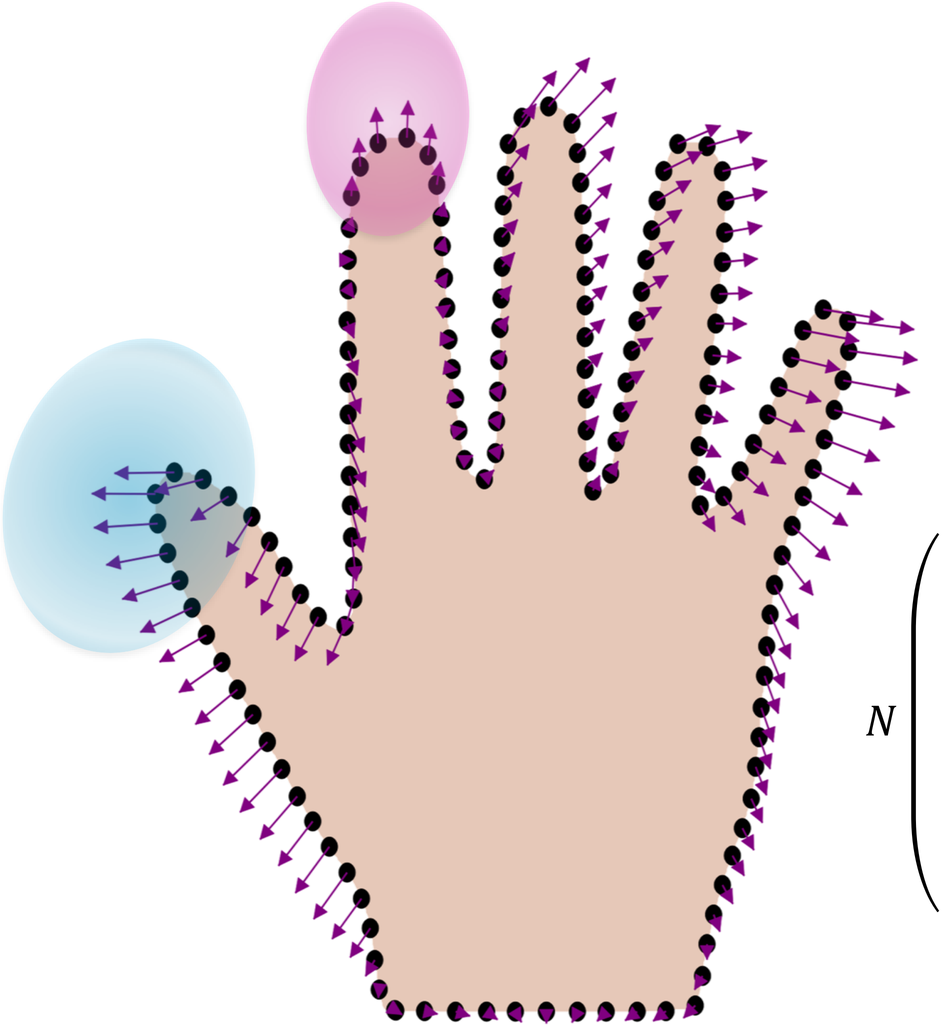
- Mean function: $\mu: \Gamma_R \to \mathbb{R}^2$
- Covariance function: $k: \Gamma_R \times \Gamma_R \to \mathbb{R}^{2\times 2}$

For any finite set Γ_R we can define $u \sim N(\vec{\mu}, K)$, with

$$\vec{\mu} = \left(\mu(x)\right)_{x\in\Gamma_R}$$

$$K = \left(k(x,x')\right)_{x,x'\in\Gamma_R}$$

Modelling possible deformations



$$
\begin{pmatrix} \vdots \\ u(x) \\ \vdots \\ u(x') \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ u_1(x) \\ u_2(x) \\ \vdots \\ u_1(x') \\ u_2(x') \\ \vdots \end{pmatrix} \sim N\left(\begin{pmatrix} \vdots \\ \mu_1(x) \\ \mu_2(x) \\ \vdots \\ \mu_1(x') \\ \mu_2(x') \\ \vdots \end{pmatrix}, \begin{pmatrix} & \vdots & & & \vdots & & \\ \cdots & \mathrm{k}_{11}(x,x) & \mathrm{k}_{12}(x,x) & \cdots & \mathrm{k}_{11}(x,x') & \mathrm{k}_{12}(x,x') & \cdots \\ & \mathrm{k}_{21}(x,x) & \mathrm{k}_{22}(x,x) & & \mathrm{k}_{21}(x,x') & \mathrm{k}_{22}(x,x') & \\ & \vdots & & \ddots & \vdots & & \\ \cdots & \mathrm{k}_{11}(x',x) & \mathrm{k}_{12}(x',x) & & \mathrm{k}_{11}(x',x') & \mathrm{k}_{12}(x',x') & \cdots \\ & \mathrm{k}_{21}(x',x) & \mathrm{k}_{22}(x',x) & \cdots & \mathrm{k}_{21}(x',x') & \mathrm{k}_{22}(x',x') & \\ & \vdots & & & \vdots & & \end{pmatrix} \right)
$$

Formal definition

A Gaussian process

$$p(u) = GP(\mu, k)$$

is a probability distribution over functions

$$u : \mathcal{X} \to \mathbb{R}^d$$

such that every finite restriction to function values

$$u_X = (u(x_1), \dots, u(x_n))$$

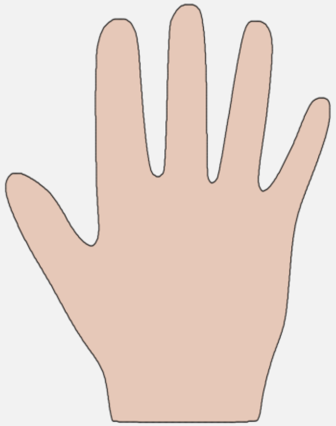
is a multivariate normal distribution

$$p(u_X) = N(\mu_X, k_{XX}).$$

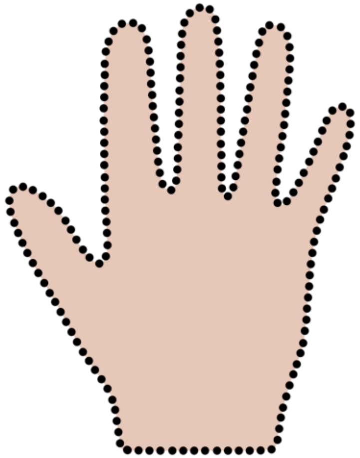
It is completely specified by a mean function μ and covariance function (or kernel) k.

Model continuously – compute discretely

Continuous representation:
$GP(\mu, k)$

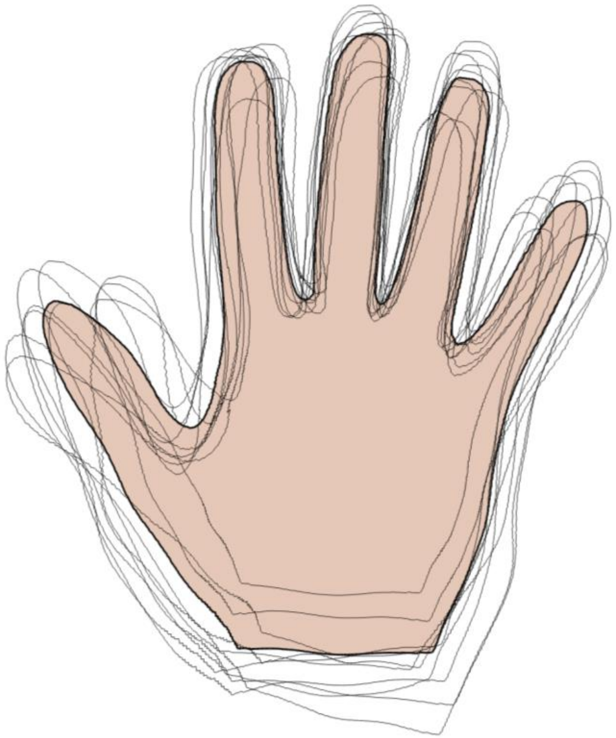


Discrete representation:
$N(\mu, K)$



The mean function

$\mu: \Gamma_R \to \mathbb{R}^d$ defines how the average deformation looks like



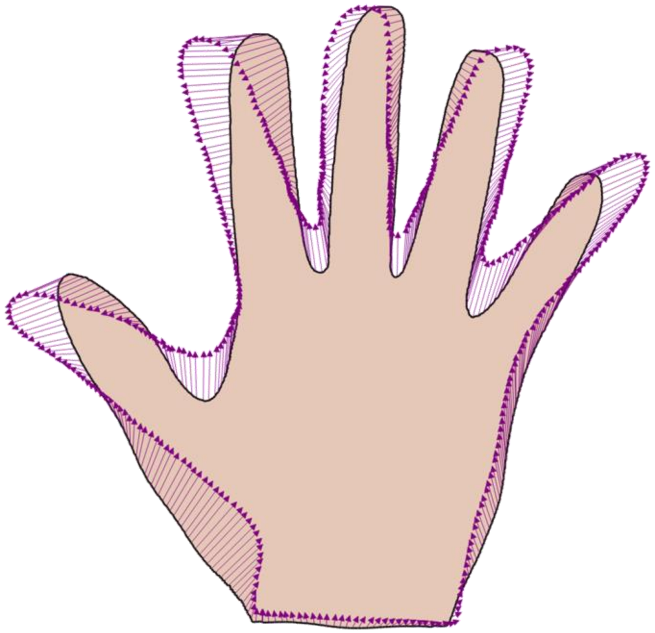
Usual assumption:

- The reference shape is an average shape.

$$\mu(x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The covariance function

$k: \Gamma_R \times \Gamma_R \rightarrow \mathbb{R}^{d \times d}$ Defines characteristics of the deformations fields



Usual assumption

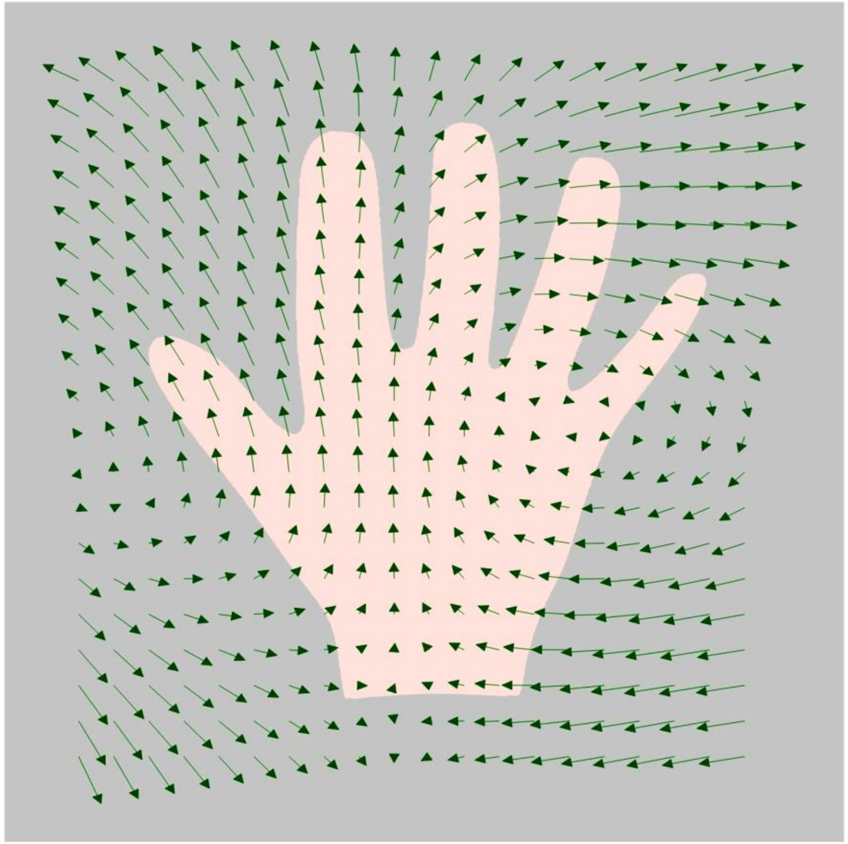
- deformation fields are smooth

Example covariance function

Squared exponential covariance function (Gaussian kernel)

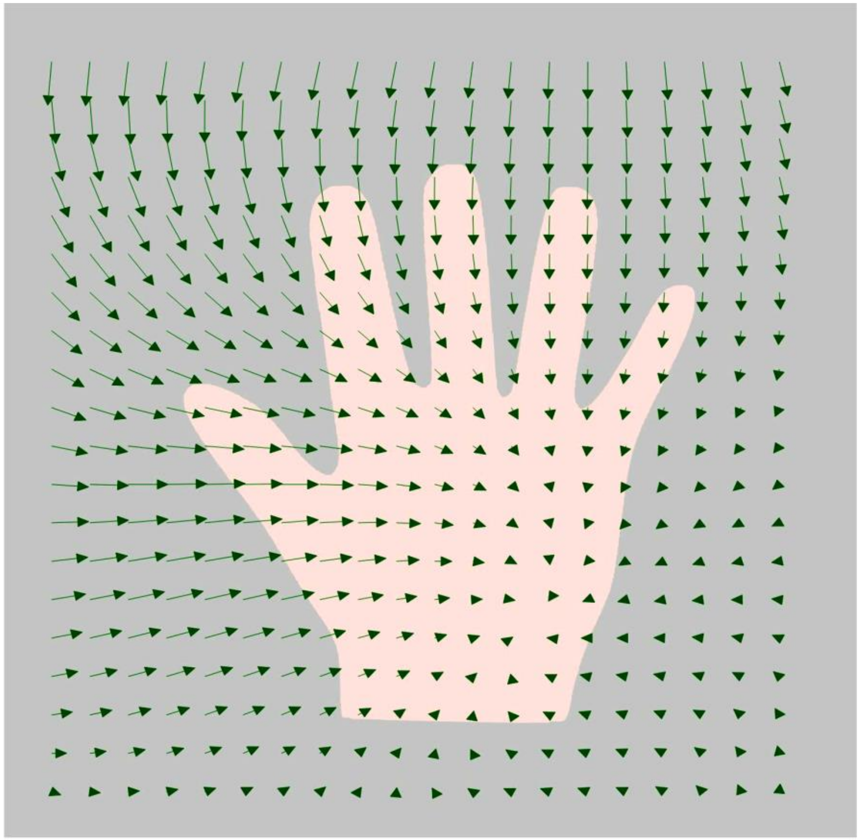
$$k(x, x') = \begin{pmatrix} s_1 \exp\left(-\frac{\|x - x'\|^2}{\sigma_1^2}\right) & 0 \\ 0 & s_2 \exp\left(-\frac{\|x - x'\|^2}{\sigma_2^2}\right) \end{pmatrix}$$

A model for smooth deformations



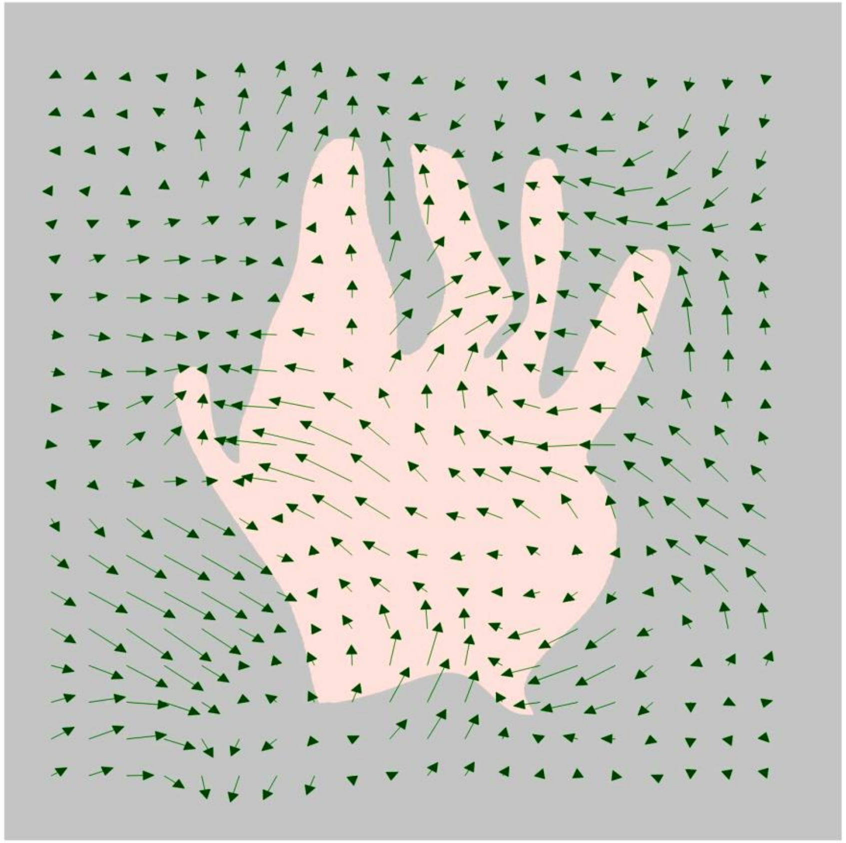
$s_1 = s_2$ large, $\sigma_1 = \sigma_2$ large

A model for smooth deformations



$s_1 = s_2$ small, $\sigma_1 = \sigma_2$ large

A model for smooth deformations



$s_1 = s_2$ large, $\sigma_1 = \sigma_2$ small

Combining kernels

Rules for constructing kernels

1. $k(x, x') = k_1(x, x') + k_2(x, x')$
2. $k(x, x') = \alpha k_1(x, x'), \alpha \in \mathbb{R}_+$
3. $k(x, x') = k_1(x, x') \odot k_2(x, x')$
4. $k(x, x') = f(x)\, f(x')^T, f: X \to \mathbb{R}^d$
5. $k(x, x') = B^T k(x, x') B, \quad \mathrm{B} \in \mathbb{R}^{r \times d}$

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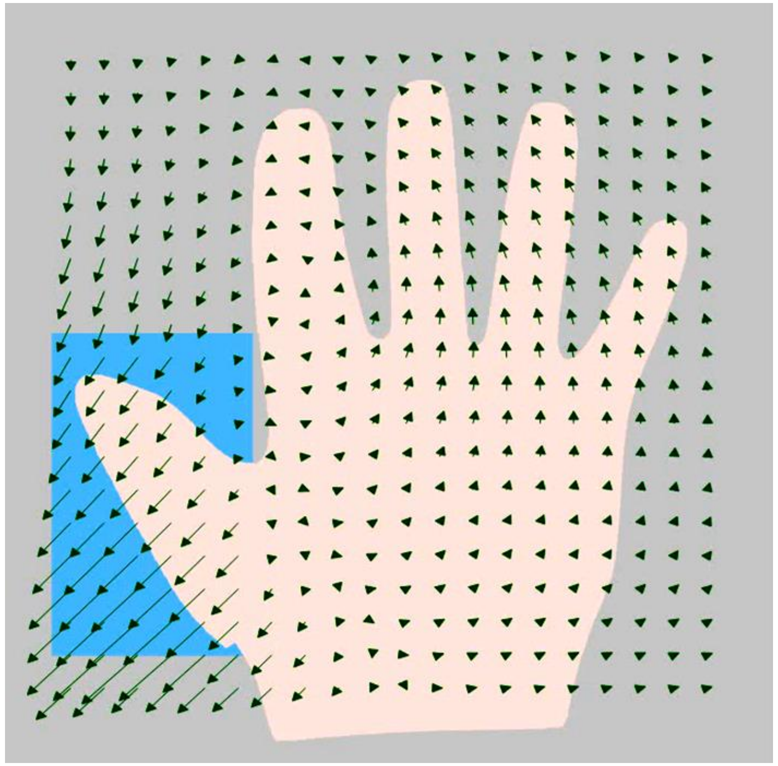
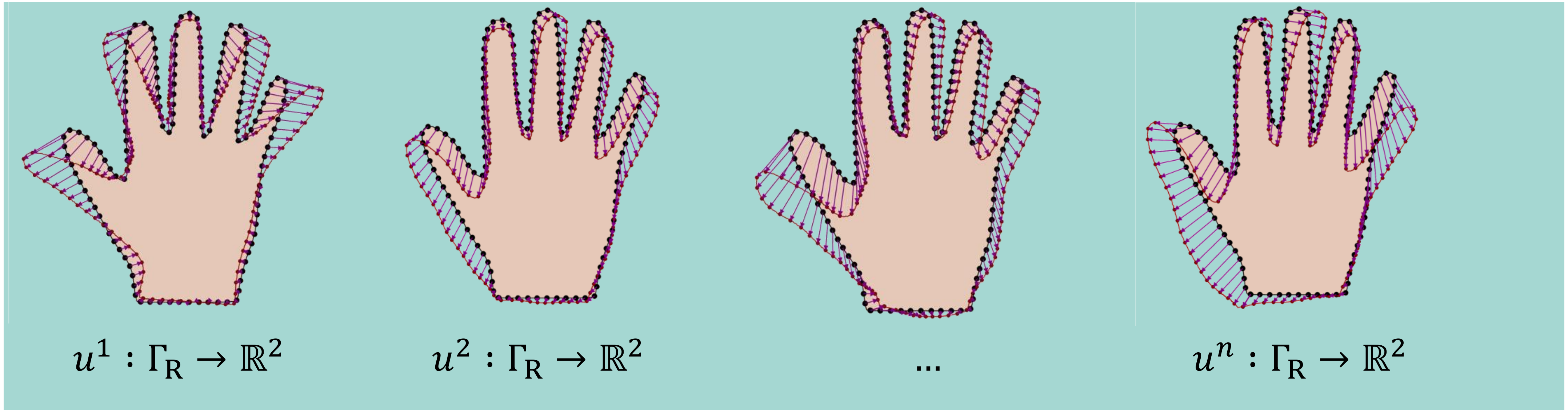


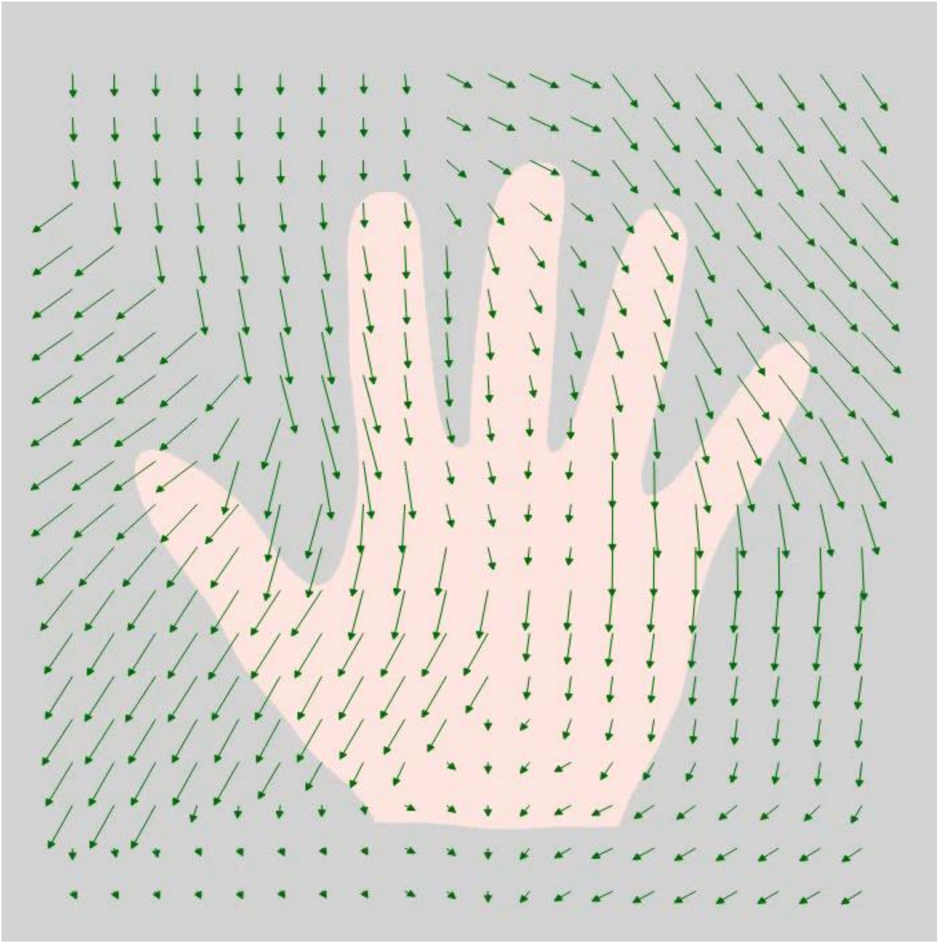
Statistical shape models

Idea: Models are learned from example deformation fields



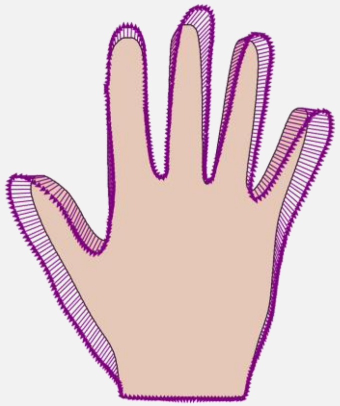
$$\mu(x) = \overline{u}(x) = \frac{1}{n}\sum_{i-1}^{n} u^i\,(x) \qquad k(x, x') = \frac{1}{n-1}\sum_{i}^{n}\left(u^i(x) - \overline{u}(x)\right)\left(u^i(x') - \overline{u}(x')\right)^T$$

Statistical shape models



Finite observations revisited

Continuous representation

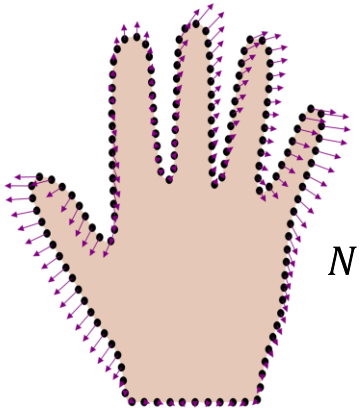


$$GP(\mu, k)$$

Too Large for real 3D shapes

tation

Discrete representation



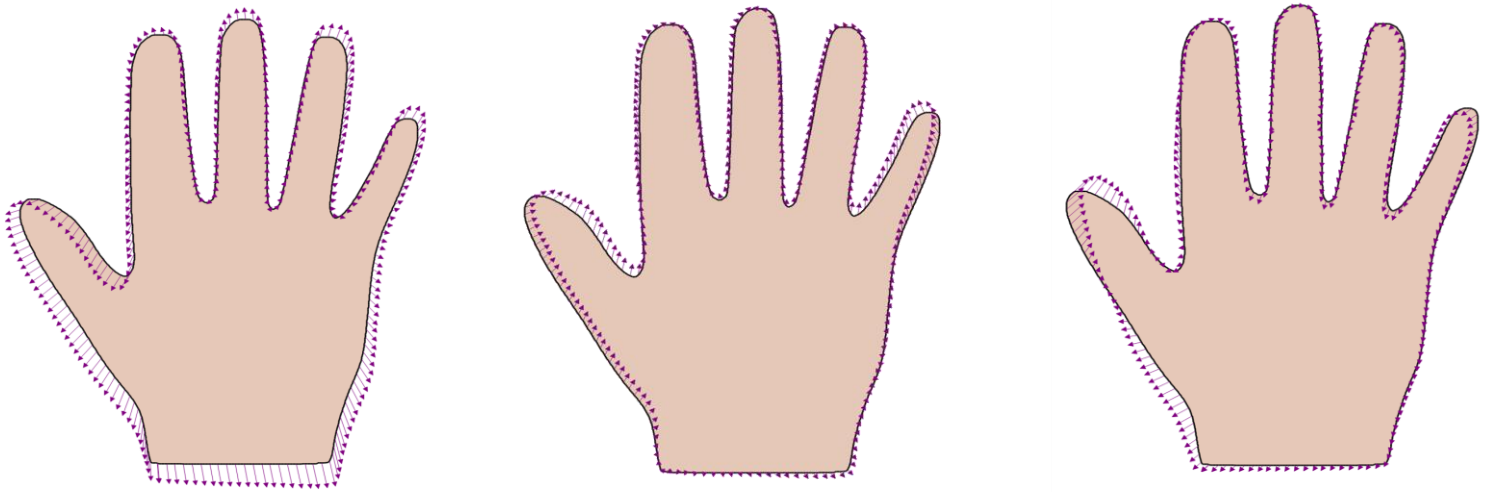
$$N\left(\begin{pmatrix}\mu_1(x_1)\\ \mu_2(x_1)\\ \vdots \\ \mu_1(x_n)\\ \mu_2(x_n)\end{pmatrix}, \begin{pmatrix} k_{11}(x_1,x_1) & k_{12}(x_1,x_1) & \cdots & k_{11}(x_1,x_n) & k_{12}(x_1,x_n)\\ k_{21}(x_1,x_1) & k_{22}(x_1,x_1) & & k_{21}(x_1,x_n) & k_{22}(x_1,x_n)\\ \vdots & & \ddots & & \vdots \\ k_{11}(x_n,x_1) & k_{12}(x_n,x_1) & \cdots & k_{11}(x_n,x_n) & k_{12}(x_n,x_n)\\ k_{21}(x_n,x_1) & k_{22}(x_n,x_1) & & k_{21}(x_n,x_n) & k_{22}(x_n,x_n)\end{pmatrix}\right)$$

The Karhunen-Loève expansion

We can write $$u \sim GP(\mu, k)$$

as $$u \sim \mu + \sum_{i=1}^{\infty} \alpha_i \sqrt{\lambda_i}\, \phi_i, \;\; \alpha_i \sim N(0,1)$$

ϕ_i is the Karhunen-Loève basis and λ_i a scaling factor



$\sqrt{\lambda_1}\phi_1$ $\sqrt{\lambda_2}\phi_2$ $\sqrt{\lambda_3}\phi_3$

Low-rank approximation

Approximation of rank r

$$u \sim \mu + \sum_{i=1}^{r} \alpha_i \sqrt{\lambda_i}\, \phi_i, \qquad \alpha_i \sim N(0,1)$$

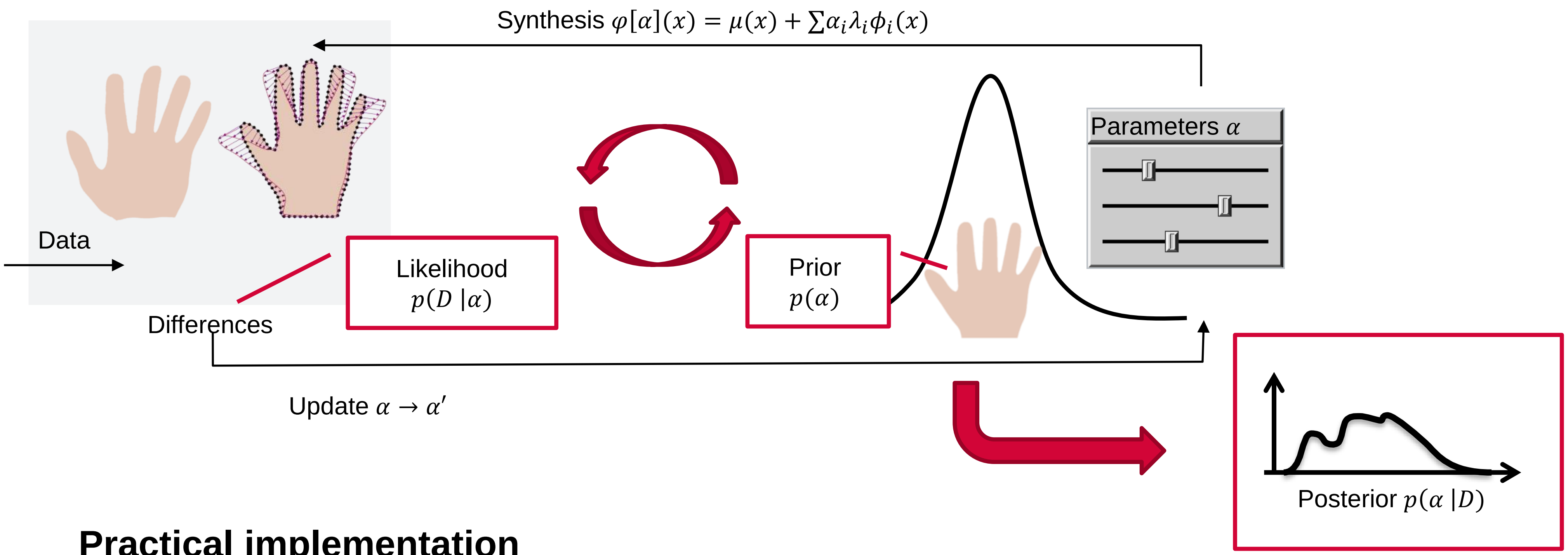
Any deformation u is determined by the coefficients
$\alpha = (\alpha_1, \dots, \alpha_r)$

$$p(u) = p(\alpha) = \prod_{i=1}^{r} \frac{1}{\sqrt{2\pi}} \exp(-\alpha_i^2/2)$$

Parametric nonparametrics

- *We use GPs as a modelling tool, and not because of infinite basis functions.*

Inference: Analysis-by-Synthesis



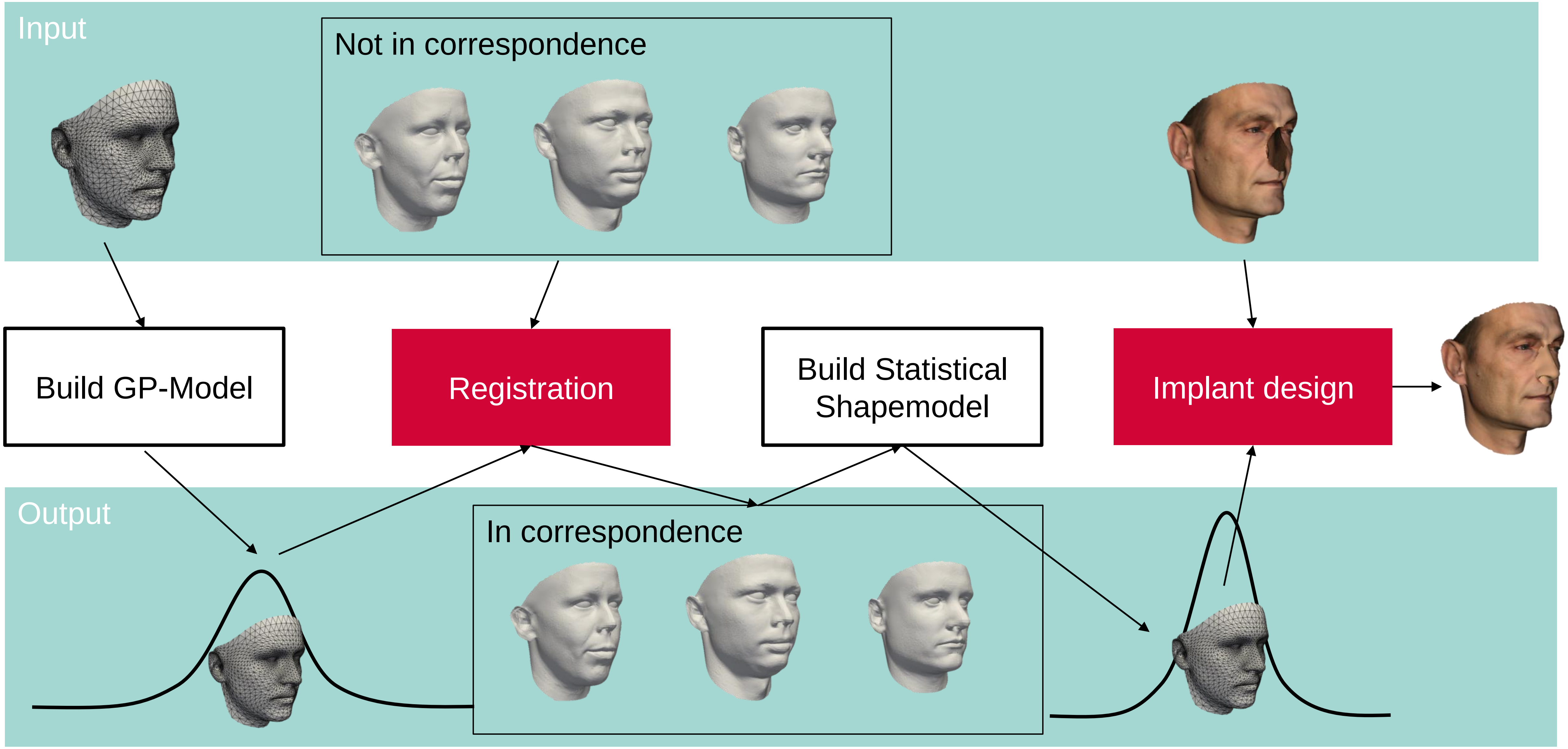
Practical implementation

- Metropolis-Hastings sampling

Demo
Designing an implant

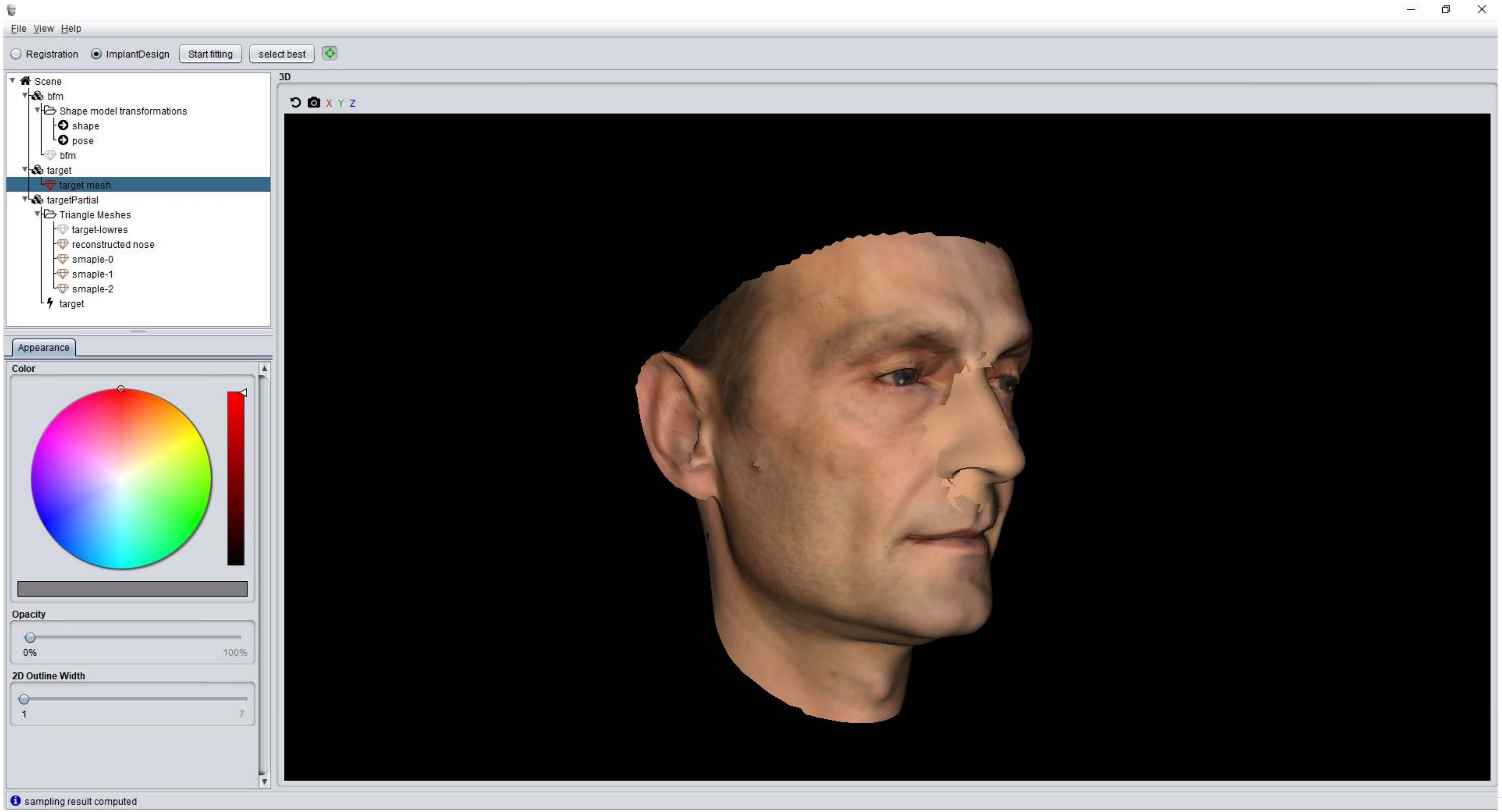
Image: Jasenko Zivanov

Workflow



Input
Not in correspondence
Build GP-Model
Registration
Build Statistical Shapemodel
Implant design
Output
In correspondence

Demo



File View Help
Registration
ImplantDesign
Start fitting
select best
Scene
bfm
Shape model transformations
shape
pose
bfm
target
target mesh
targetPartial
Triangle Meshes
target-lowres
reconstructed nose
smaple-0
smaple-1
smaple-2
target
3D
X Y Z
Appearance
Color
Opacity
0%
100%
2D Outline Width
1
7
sampling result computed

References

Contact
marcel.luethi@unibas.ch

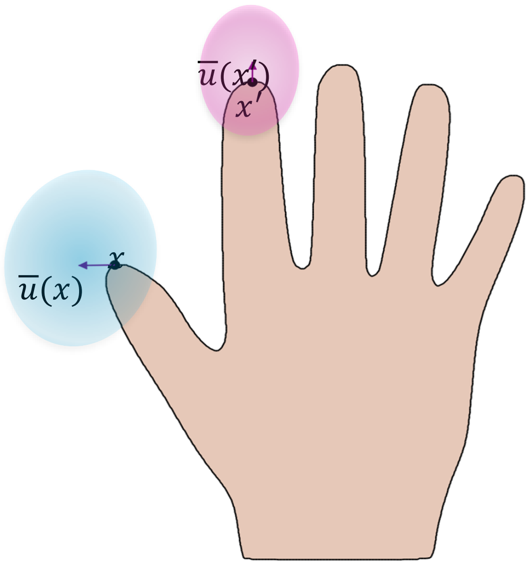
Graphics and Vision Research Group
http://gravis.dmi.unibas.ch/

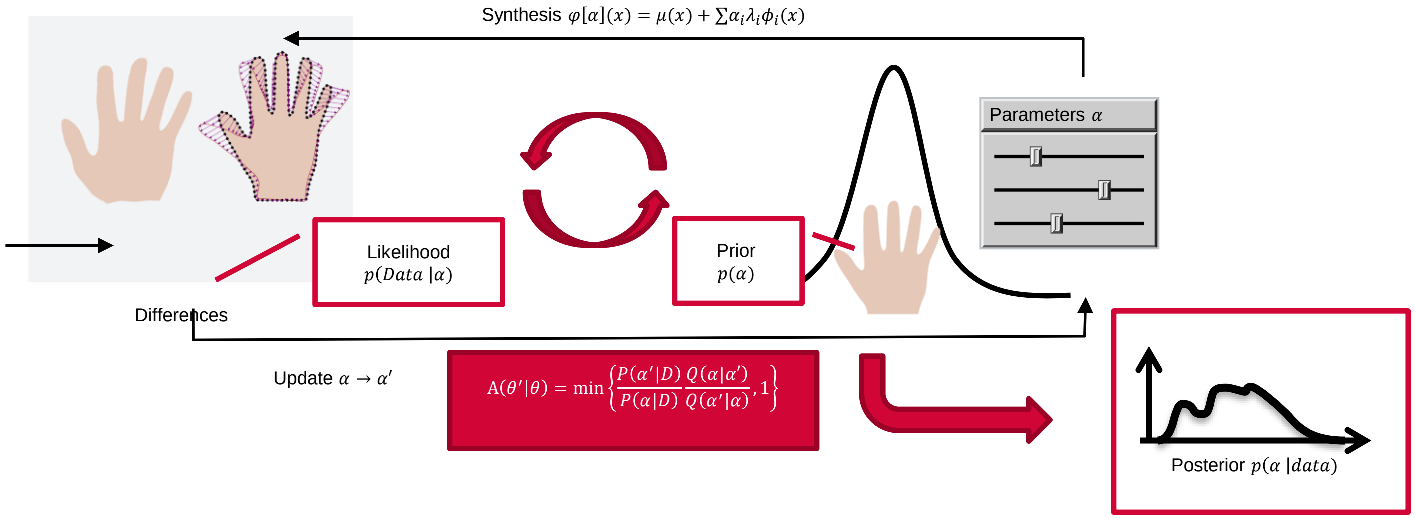
Open Source Software
www.scalismo.org

Theory

- Lüthi, M., Gerig, T., Jud, C., & Vetter, T. (2017). Gaussian process morphable models. *IEEE transactions on pattern analysis and machine intelligence*
- Schönborn, S., Egger, B., Morel-Forster, A., & Vetter, T. (2017). Markov chain monte carlo for automated face image analysis. *International Journal of Computer Vision*, *123*(2), 160-183.

Questions?


$\bar{u}(x')$
x'
$\bar{u}(x)$
x


Synthesis $\varphi[\alpha](x) = \mu(x) + \sum \alpha_i \lambda_i \phi_i(x)$
Parameters α
Likelihood $p(Data\,|\alpha)$
Prior $p(\alpha)$
Differences
Update $\alpha \rightarrow \alpha'$
$A(\theta'|\theta) = \min\left\{\frac{P(\alpha'|D)\;Q(\alpha|\alpha')}{P(\alpha|D)\;Q(\alpha'|\alpha)}, 1\right\}$
Posterior $p(\alpha\,|data)$